

Spin Distributions for Bipartite Quantum Systems

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We carryout a comparative study of spin distributions defined over the sphere for bipartite quantum spin assemblies. We analyse Einstein-Podolsky-Rosen-Bohm (EPRB) spin correlations in a spin- s singlet state using these distributions. We observe that in the classical limit of $s \rightarrow \infty$, EPRB spin distributions turn out to be delta functions, thus reflecting the perfect anticorrelation property of two spin vectors associated with a spin- s singlet state.

I. INTRODUCTION

Distribution function description of quantum mechanics owes its origin to the work of Wigner [1]. This description offers a framework in which quantum phenomena can be discussed using as much classical language as allowed. It appeals naturally to one's intuition and can often provide useful physical insight that can not be easily gained through other approaches [2]. Irrespective of a few shortcomings, like for *eg.*, appearance of negative probabilities, advantage of using the distribution function approach is in the fact that it involves classical functions as opposed to operators. Besides the phase space distribution functions, where position q and momentum p are the statistical variables, there have been several attempts to construct the spin distribution functions([3]-[6]). Infact, *Wigner like* distribution functions have been used ([5], [7]) to discuss EPRB spin correlations in a spin- s singlet state. In this formulation, the spin correlations are cast in a structurally similar form to those of local hidden variable models but with negative distribution functions.

In this paper we extend the spin distribution functions defined over the sphere viz., the P -, Q -, and F - functions ([4],[6]) for bipartite spin systems characterized by a mixed state spin density matrix $\hat{\rho}$. In Section II we discuss the expansion of the spin density matrix in terms of Fano statistical tensor parameters [8]. In Section III we express the P -, Q -, and F - spin distributions in terms of Fano statistical tensor parameters. In Section IV we extend these distributions to bipartite spin systems and study the EPRB spin correlations using them. We also show that distribution functions characterizing EPRB spin- s singlet state approach delta function in the classical limit of $s \rightarrow \infty$.

II. FANO STATISTICAL TENSOR PARAMETERS

A set of $(2s+1)^2$ spherical tensor operators $\{\hat{\tau}_q^k(\hat{\vec{S}}), k = 0, 1, 2, \dots, 2s \text{ and } q = -k, -k+1, \dots, k\}$ constructed out of the spin operator $\hat{\vec{S}}$ through [9]

$$\begin{aligned} \hat{\tau}_q^k(\hat{\vec{S}}) &= \mathcal{N}_{sk} \left(\hat{\vec{S}} \cdot \vec{\nabla} \right)^k \left\{ r^k Y_{kq}(\theta, \phi) \right\}; \\ \mathcal{N}_{sk} &= \frac{2^k}{k!} \left[\frac{4\pi (2s-k)! (2s+1)}{(2s+k+1)!} \right]^{\frac{1}{2}}, \end{aligned} \quad (2.1)$$

provides a linearly independent, orthonormal basis and any operator, acting on the $(2s+1)$ dimensional Hilbert space of a spin- s assembly, could be resolved into its irreducible components in terms of this basis. Here, $Y_{kq}(\theta, \phi)$ denote spherical harmonic functions and the normalisation constants \mathcal{N}_{sk} are chosen so as to be consistent with the Madison convention [10]:

$$\langle sm' | \hat{\tau}_q^k(\hat{\vec{S}}) | sm \rangle = \sqrt{2k+1} c(sks; mqm'), \quad (2.2)$$

where $c(sks; mqm')$ denote Clebsch-Gordan coefficients. From the property

$$\sum_{m=-s}^s c(sks; mqm) = (2s+1) \delta_{k0} \delta_{q0} \quad (2.3)$$

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of the Clebsch-Gordan coefficients, it could be easily verified that $\hat{\tau}_q^k(\hat{\vec{S}})$ are traceless for all non zero values of k and $\hat{\tau}_0^0(\hat{\vec{S}})$ is a $(2s+1) \times (2s+1)$ dimensional unit matrix.

By construction, $\hat{\tau}_q^k(\hat{\vec{S}})$ are irreducible under rotations, with the transformation property:

$$\begin{aligned} \hat{\tau}_q^k(\hat{\vec{S}}) \xrightarrow{\text{rotation}} \left[\hat{\tau}_q^k(\hat{\vec{S}}) \right]^{\text{rotated}} &= \hat{R}(\alpha, \beta, \gamma) \hat{\tau}_q^k(\hat{\vec{S}}) \hat{R}^\dagger(\alpha, \beta, \gamma) \\ &= \sum_{q'=-k}^k D_{q'q}^k(\alpha, \beta, \gamma) \hat{\tau}_{q'}^k(\hat{\vec{S}}), \end{aligned} \quad (2.4)$$

where D^k denotes $(2k+1)$ dimensional irreducible representation of rotations [11] and α, β, γ are the Euler angles of rotation. Hermiticity property of $\hat{\tau}_q^k(\hat{\vec{S}})$ is expressed through,

$$\hat{\tau}_q^k(\hat{\vec{S}})^\dagger = (-1)^q \hat{\tau}_{-q}^k(\hat{\vec{S}}). \quad (2.5)$$

The orthonormality property,

$$\text{Tr} \left(\hat{\tau}_q^k(\hat{\vec{S}}) \hat{\tau}_{q'}^{k'}(\hat{\vec{S}})^\dagger \right) = (2s+1) \delta_{kk'} \delta_{qq'}, \quad (2.6)$$

could be readily realised by making use of Eq.(2.2), (2.5) and the properties,

$$\begin{aligned} c(s_1 s_2 s; m_1 m_2 m) &= (-1)^{s_1 - m_1} \sqrt{\frac{2s+1}{2s_2+1}} c(s_1 s_2 s; m_1 - m - m_2), \\ c(s_1 s_2 s; m_1 m_2 m) &= (-1)^{s_1 + s_2 - s} c(s_1 s_2 s; -m_1 - m_2 - m), \\ c(s_1 s_2 s; m_1 m_2 m) &= (-1)^{s_1 + s_2 - s} c(s_2 s_1 s; m_2 m_1 m), \end{aligned} \quad (2.7)$$

together with the orthogonality

$$\sum_{m_1=-s_1}^{s_1} \sum_{m_2=-s_2}^{s_2} c(s_1 s_2 s; m_1 m_2 m) c(s_1 s_2 s'; m_1 m_2 m') = \delta_{ss'} \delta_{mm'}, \quad (2.8)$$

of the Clebsch-Gordan coefficients.

Any arbitrary operator $\hat{A}(\hat{\vec{S}})$ can be expressed in terms of the irreducible tensor operators $\hat{\tau}_q^k(\hat{\vec{S}})$ as

$$\hat{A} = \sum_{k=0}^{2s} \sum_{q=-k}^k \hat{\tau}_q^k(\hat{\vec{S}})^\dagger a_q^k \quad (2.9)$$

where the spherical componets a_q^k of the operator \hat{O} are given by

$$a_q^k = \text{Tr} \left(\hat{O} \hat{\tau}_q^k(\hat{\vec{S}}) \right). \quad (2.10)$$

Specifically, the spin density matrix $\hat{\rho}$, which characterises a spin- s assembly, has the resolution:

$$\hat{\rho} = \frac{1}{2s+1} \sum_{k=0}^{2s} \sum_{q=-k}^k \hat{\tau}_q^k(\hat{\vec{S}})^\dagger t_q^k, \quad (2.11)$$

and $t_q^k = \langle \hat{\tau}_q^k(\hat{\vec{S}}) \rangle = \text{Tr} \left(\hat{\rho} \hat{\tau}_q^k(\hat{\vec{S}}) \right)$ are the well-known Fano statistical tensor parameters[8].

Depending on the requirements of the spin density matrix $\hat{\rho}$, the Fano statistical tensor parameters t_q^k satisfy the following properties:

- Normalization: $\text{Tr}(\hat{\rho}) = 1 \implies t_0^0 = 1$,
- Hermiticity: $\hat{\rho}^\dagger = \hat{\rho} \implies t_q^{k*} = (-1)^q t_{-q}^k$,
- Property under rotation:

$$\begin{aligned}
(\hat{\rho})^{\text{rotated}} &= \frac{1}{2s+1} \sum_{k,q} \left(\hat{\tau}_q^k (\hat{\vec{S}})^\dagger \right)^{\text{rotated}} t_q^k \\
&= \frac{1}{2s+1} \sum_{k,q} \hat{\tau}_q^k (\hat{\vec{S}})^\dagger (t_q^k)^{\text{rotated}} \\
&\Rightarrow (t_q^k)^{\text{rotated}} = \sum_{q'=-k}^k D_{q'q}^k(\alpha, \beta, \gamma) t_{q'}^k.
\end{aligned}$$

For an entangled spin assembly containing a pair of subsystems with spins s_1 and s_2 , the above discussion can be readily generalised ([12], [13]), and the spin density matrix characterising such a composite bipartite system is given by,

$$\hat{\rho}_{12} = \frac{1}{(2s_1+1)(2s_2+1)} \sum_{k_1=0}^{2s_1} \sum_{q_1=-k_1}^{k_1} \sum_{k_2=0}^{2s_2} \sum_{q_2=-k_2}^{k_2} t_{q_1 q_2}^{k_1 k_2*} \left(\hat{\tau}_{q_1}^{k_1} (\hat{\vec{S}}_1) \otimes \hat{\tau}_{q_2}^{k_2} (\hat{\vec{S}}_2) \right), \quad (2.12)$$

where $t_{q_1 q_2}^{k_1 k_2}$ denote coupled Fano statistical tensor parameters, $\hat{\vec{S}}_1, \hat{\vec{S}}_2$ denote the spin operators of the subsystems 1 and 2 respectively. The reduced or the subsystem density matrices are obtained by taking the partial traces:

$$\hat{\rho}_1 = \text{Tr}_2 (\hat{\rho}_{12}) = \frac{1}{(2s_1+1)} \sum_{k_1=0}^{2s_1} \sum_{q_1=-k_1}^{k_1} \hat{\tau}_{q_1}^{k_1} (\hat{\vec{S}}_1)^\dagger t_{q_1 0}^{k_1 0}, \quad (2.13)$$

and

$$\hat{\rho}_2 = \text{Tr}_1 (\hat{\rho}_{12}) = \frac{1}{(2s_2+1)} \sum_{k_2=0}^{2s_2} \sum_{q_2=-k_2}^{k_2} \hat{\tau}_{q_2}^{k_2} (\hat{\vec{S}}_2)^\dagger t_{0 q_2}^{0 k_2}. \quad (2.14)$$

where we have made use of Eq.(2.2) and (2.3). The system is said to be entangled iff

$$\hat{\rho}_{12} \neq \hat{\rho}_1 \otimes \hat{\rho}_2 \Rightarrow t_{q_1 q_2}^{k_1 k_2} \neq t_{q_1 0}^{k_1 0} t_{0 q_2}^{0 k_2}. \quad (2.15)$$

III. $P - Q -$ AND F DISTRIBUTIONS IN TERMS OF FANO STATISTICAL TENSOR PARAMETERS.

The coherent state description of electromagnetic fields has proved to be successful in providing an insight into the relationship between semiclassical and quantum theories of light [14]. The diagonal coherent state representation or P -representation for the density matrix,

$$\hat{\rho} = \int d\alpha P(\alpha) |\alpha\rangle \langle \alpha|, \quad (3.1)$$

has proven to be very useful in bringing many of the results of quantum electrodynamics into forms similar to those of classical theory and the expectation values of any quantum operators \hat{A} could be realised as classical averages in terms of the weight function $P(\alpha)$ through

$$\langle \hat{A} \rangle = \text{Tr} [\hat{\rho} \hat{A}] = \int d\alpha P(\alpha) A(\alpha), \quad (3.2)$$

where $A(\alpha)$ is the classical function corresponding to the operator \hat{A} . Arecchi *et.al.*, [4] introduced the analogue of the diagonal coherent state representation for the spin density matrix $\hat{\rho}$ via the relation

$$\hat{\rho} = \int d\Omega P(\theta, \phi) |\theta\phi\rangle \langle \theta\phi|, \quad (3.3)$$

where $d\Omega = \sin\theta d\theta d\phi$ and $|\theta\phi\rangle$ represents the spin coherent state(SCS) or Bloch state, defined as a rotated maximum 'down' spin state $|s-s\rangle$:

$$\begin{aligned}
|\theta\phi\rangle &= e^{(\tau\hat{S}_+ - \tau^*\hat{S}_-)} |s-s\rangle = \hat{R}(\phi - \pi, \theta, \pi - \phi) |s-s\rangle \\
&= \sum_{m=-s}^s \sqrt{\binom{2s}{s+m}} \left(\cos\frac{\theta}{2}\right)^{s-m} \left(\sin\frac{\theta}{2}\right)^{s+m} e^{-i(s+m)\phi} |sm\rangle.
\end{aligned} \tag{3.4}$$

Here, $\tau = \frac{1}{2}\theta e^{-i\phi}$ and \hat{S}_{\pm} are the spin ladder operators; $\hat{R}(\phi - \pi, \theta, \pi - \phi)$ denotes rotation through Euler angles $\phi - \pi$, θ and $\pi - \phi$. From the normalisation condition $\text{Tr}[\hat{\rho}] = 1$ we have

$$\int d\Omega P(\theta, \phi) = 1 \tag{3.5}$$

i.e., the weight function $P(\theta, \phi)$ in the diagonal spin coherent state representation is a normalised function. With the help of $P(\theta, \phi)$, the quantum expectation value of any arbitrary spin observable \hat{A} is given by the classical average

$$\langle \hat{A} \rangle = \int d\Omega P(\theta, \phi) A(\theta, \phi), \tag{3.6}$$

where

$$A(\theta, \phi) = \langle \theta\phi | \hat{A} | \theta\phi \rangle, \tag{3.7}$$

is a classical function corresponding to the quantum mechanical operator \hat{A} . For example corresponding to the spin operator $\hat{S} = (\hat{S}_1, \hat{S}_2, \hat{S}_3)$ we obtain

$$\vec{S}(\theta, \phi) = \langle \theta\phi | \hat{S} | \theta\phi \rangle = s \vec{n}(\pi - \theta, \phi), \tag{3.8}$$

where $\vec{n}(\pi - \theta, \phi) \equiv (\sin\theta \cos\phi, \sin\theta \sin\phi, -\cos\theta)$ is a unit vector defined on the Bloch sphere. Thus in the P -representation of spin, one can visualise the quantum expectation value of spin as a classical statistical average of random orientations of angular directions $\vec{n}(\pi - \theta, \phi)$.

Observe that the Fano statistical tensors t_q^k can be expressed as the classical averages

$$t_q^k = \text{Tr} \left[\hat{\rho} \hat{\tau}_q^k(\hat{S}) \right] = \int d\Omega P(\theta, \phi) \langle \theta\phi | \hat{\tau}_q^k(\hat{S}) | \theta\phi \rangle. \tag{3.9}$$

The expectation values of the irreducible tensor operators $\hat{\tau}_q^k(\hat{S})$ in the spin coherent states $|\theta\phi\rangle$ can be simplified as follows:

$$\begin{aligned}
\langle \theta\phi | \hat{\tau}_q^k(\hat{S}) | \theta\phi \rangle &= \langle s-s | \hat{R}^\dagger(\phi - \pi, \theta, \pi - \phi) \hat{\tau}_q^k(\hat{S}) \hat{R}(\phi - \pi, \theta, \pi - \phi) | s-s \rangle \\
&= \sum_{q'=-k}^k \langle s-s | \hat{\tau}_{q'}^k(\hat{S}) | s-s \rangle D_{qq'}^{k*}(\phi - \pi, \theta, \pi - \phi) \\
&= \sum_{q'=-k}^k c(ks; -s0 - s) \sqrt{2k+1} \delta_{q'0} D_{qq'}^{k*}(\phi - \pi, \theta, \pi - \phi) \\
&= (-1)^k c(ks; s0s) \sqrt{4\pi} Y_{kq}(\theta, \phi - \pi),
\end{aligned} \tag{3.10}$$

where we have made use of the transformation property of $\hat{\tau}_q^k(\hat{S})$ under rotations. Utilising the explicit expression for the Clebsch-Gordan coefficient $c(ks; s0s)$ [15] given by,

$$c(ks; s0s) = (2s)! \left[\frac{(2s+1)}{(2s-k)!(2s+k+1)!} \right]^{\frac{1}{2}}, \tag{3.11}$$

we obtain,

$$\langle \theta\phi | \hat{\tau}_q^k(\hat{S}) | \theta\phi \rangle = \sqrt{4\pi} (-1)^{k+q} (2s)! \left[\frac{(2s+1)}{(2s-k)!(2s+k+1)!} \right]^{\frac{1}{2}} Y_{kq}(\theta, \phi), \tag{3.12}$$

where we have used the symmetry [11] $Y_{kq}(\theta, \phi - \pi) = (-1)^q Y_{kq}(\theta, \phi)$, of the spherical harmonic functions. Substituting Eq.(3.12) in Eq.(3.9), we observe that $P(\theta, \phi)$ must be of the form

$$P(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \sum_{k=0}^{2s} \sum_{q=-k}^k (-1)^{k+q} \mathcal{P}_{sk} t_q^k Y_{kq}^*(\theta, \phi), \quad (3.13)$$

in order to reproduce the Fano statistical tensors t_q^k in this representation. The coefficients \mathcal{P}_{sk} are given by,

$$\mathcal{P}_{sk} = \frac{1}{(2s)!} \sqrt{\frac{(2s-k)!(2s+k+1)!}{(2s+1)!}}. \quad (3.14)$$

Another useful distribution function which can be derived using SCS is the positive, normalised Q -function[2, 5]:

$$Q(\theta, \phi) = \frac{(2s+1)}{4\pi} \langle \theta\phi | \hat{\rho} | \theta\phi \rangle. \quad (3.15)$$

Using Eq.(2.11) and (3.10) we obtain,

$$\begin{aligned} Q(\theta, \phi) &= \frac{(2s+1)}{4\pi} \langle \theta\phi | \hat{\rho} | \theta\phi \rangle = \frac{1}{4\pi} \sum_{k=0}^{2s} \sum_{q=-k}^k t_q^{k*} \langle \theta\phi | \hat{\tau}_q^k(\hat{S}) | \theta\phi \rangle \\ &= \frac{1}{4\pi} \sum_{k=0}^{2s} \sum_{q=-k}^k \sqrt{4\pi} t_q^{k*} (-1)^{k+q} Y_{kq}(\theta, \phi) (2s)! \sqrt{\frac{(2s+1)}{(2s-k)!(2s+k+1)!}} \\ &= \frac{1}{\sqrt{4\pi}} \sum_{k=0}^{2s} \sum_{q=-k}^k (-1)^{k+q} \mathcal{Q}_{sk} t_q^k Y_{kq}^*(\theta, \phi) \end{aligned} \quad (3.16)$$

where

$$\mathcal{Q}_{sk} = (2s)! \sqrt{\frac{(2s+1)}{(2s-k)!(2s+k+1)!}}. \quad (3.17)$$

It could be readily verified, using the orthonormality property of the spherical harmonics[11],

$$\int d\Omega Y_{kq}(\theta, \phi) Y_{k'q'}(\theta, \phi) = \delta_{k, k'} \delta_{q, q'}, \quad (3.18)$$

that the P - and Q - functions [16] given by Eq.(3.13) and Eq.(3.16) respectively, are normalised:

$$\begin{aligned} \int d\Omega P(\theta, \phi) &= \frac{1}{\sqrt{4\pi}} \sum_{k=0}^{2s} \sum_{q=-k}^k (-1)^{k+q} \mathcal{P}_{sk} t_q^k \int d\Omega Y_{kq}^*(\theta, \phi) \\ &= \frac{1}{\sqrt{4\pi}} \sum_{k=0}^{2s} \sum_{q=-k}^k (-1)^{k+q} \mathcal{P}_{sk} t_q^k \sqrt{4\pi} \delta_{k, 0} \delta_{q, 0} = 1 \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} \int d\Omega Q(\theta, \phi) &= \frac{1}{\sqrt{4\pi}} \sum_{k=0}^{2s} \sum_{q=-k}^k (-1)^{k+q} \mathcal{Q}_{sk} t_q^k \int d\Omega Y_{kq}^*(\theta, \phi) \\ &= \frac{1}{\sqrt{4\pi}} \sum_{k=0}^{2s} \sum_{q=-k}^k (-1)^{k+q} \mathcal{Q}_{sk} t_q^k \sqrt{4\pi} \delta_{k, 0} \delta_{q, 0} = 1 \end{aligned} \quad (3.20)$$

Also, the reality of the sums $\sum_{q=-k}^k (-1)^q t_q^k Y_{kq}^*(\theta, \phi)$ ensures that $P(\theta, \phi)$ and $Q(\theta, \phi)$ are real.

One can also construct a spin distribution function over the sphere using the characteristic function approach[6]. In this method, a classical distribution function $F(\vec{X})$ with $\vec{X} \equiv (X_1, X_2, X_3)$ as the associated random variables, is constructed by taking the Fourier inverse of its characteristic function $\phi(\vec{I})$, which is the expectation value of $e^{i \vec{I} \cdot \vec{X}}$ i.e.,

$$\phi(\vec{I}) = E \left(e^{i\vec{I} \cdot \vec{X}} \right) = \int \int \int d^3 X F(\vec{X}) e^{i\vec{I} \cdot \vec{X}}. \quad (3.21)$$

Here, $E(\dots)$ denotes the expectation value. It can be readily seen that the distribution function $F(\vec{X})$ is obtained as Fourier inverse of the characteristic function $\phi(\vec{I})$:

$$F(\vec{X}) = \frac{1}{(2\pi)^3} \int \int \int d^3 I e^{-i\vec{I} \cdot \vec{X}} \phi(\vec{I}). \quad (3.22)$$

Making use of the well-known expansion

$$e^{-i\vec{I} \cdot \vec{X}} = 4\pi \sum_{k=0}^{\infty} \sum_{q=-k}^k i^k j_k(IX) Y_{kq}^*(\theta, \phi) Y_{kq}(\theta_I, \phi_I), \quad (3.23)$$

where, j_k denotes the spherical Bessel function and the spherical polar co-ordinates of \vec{I} and \vec{X} are denoted, respectively, by (I, θ_I, ϕ_I) , (X, θ, ϕ) . One can express the characteristic function $\phi(\vec{I})$ in terms of the spherical moments

$$\mu_q^k = E [j_k(IX) Y_{kq}^*(\theta, \phi)], \quad (3.24)$$

as,

$$\phi(\vec{I}) = 4\pi \sum_{k=0}^{\infty} \sum_{q=-k}^k (i)^k Y_{kq}^*(\theta_I, \phi_I) \mu_q^k. \quad (3.25)$$

In the case of quantum spin assemblies, \vec{X} corresponds to spin operators $\hat{\vec{S}} \equiv (\hat{S}_1, \hat{S}_2, \hat{S}_3)$ and the classical average $E(\dots)$ is replaced by the quantum mechanical average $\text{Tr}(\hat{\rho} \dots)$. Moreover, a spin distribution is expected to automatically reflect the constancy of the squared angular momentum i.e., $\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2 = s(s+1)$. On imposing the condition $X_1^2 + X_2^2 + X_3^2 = R^2 = s(s+1)$ and introducing solid harmonics $\mathcal{Y}_{kq}(\vec{X}) = R^k Y_{kq}(\theta, \phi)$ in Eq.(3.24) we obtain,

$$\mu_q^k = j_k(IR) R^{-k} E \left(\mathcal{Y}_{kq}(\vec{X}) \right), \quad (3.26)$$

where $j_k(IR) R^{-k}$ has been taken outside the expectation value since $R = \sqrt{s(s+1)}$ is a constant. We can now use the correspondence rule [17]

$$\mathcal{Y}_{kq}(\vec{X}) \longrightarrow \frac{1}{k! \mathcal{N}_{sk}} \hat{\tau}_q^k(\hat{\vec{S}}), \quad (3.27)$$

so that

$$E \left(\mathcal{Y}_{kq}(\vec{X}) \right) = \frac{1}{k! \mathcal{N}_{sk}} \text{Tr} \left[\hat{\rho} \hat{\tau}_q^k(\hat{\vec{S}}) \right] = \frac{t_q^k}{k! \mathcal{N}_{sk}}, \quad (3.28)$$

for $k \leq 2s$ and $E \left(\mathcal{Y}_{kq}(\vec{X}) \right) = 0$ for $k \geq 2s$, in accordance with the Wigner-Eckart theorem[11]. Thus the characteristic function of Eq.(3.25) takes the form,

$$\phi(\vec{I}) = 4\pi \sum_{k=0}^{2s} \sum_{q=-k}^k (i)^k \frac{1}{k! \mathcal{N}_{sk}} R^{-k} j_k(IR) Y_{kq}^*(\theta_I, \phi_I) t_q^k. \quad (3.29)$$

The Fourier transform of $\phi(\vec{I})$ can now be readily obtained by making use of the result[15],

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} I^2 dI \sin \theta_I d\theta_I d\phi_I e^{-i\vec{I} \cdot \vec{X}} j_k(IR) Y_{kq}^*(\theta_I, \phi_I) = \frac{2\pi^2}{R^2} (-i)^k \delta(R - X) Y_{kq}^*(\theta, \phi) \quad (3.30)$$

and we obtain the spin distribution function over the surface of the sphere of radius $R = \sqrt{s(s+1)}$:

$$\begin{aligned}
F(\vec{X}) &= \frac{1}{(2\pi)^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} I^2 dI \sin \theta_I d\theta_I d\phi_I e^{-i\vec{I} \cdot \vec{X}} \phi(\vec{I}) \\
&= \delta(R - X) \sum_{k=0}^{2s} \sum_{q=-k}^k \frac{1}{k! \mathcal{N}_{sk}} R^{-k-2} t_q^k Y_{kq}^*(\theta, \phi).
\end{aligned} \tag{3.31}$$

The normalised distribution $F(\theta, \phi)$ of angular co-ordinates θ, ϕ is obtained through the relation,

$$F(\theta, \phi) = \int_0^\infty X^2 dX F(\vec{X}) = \sum_{k=0}^{2s} \sum_{q=-k}^k \mathcal{F}_{sk} t_q^k Y_{kq}^*(\theta, \phi) \tag{3.32}$$

where

$$\mathcal{F}_{sk} = \frac{1}{2^k} \left[\frac{(2s+k+1)!}{4\pi(2s-k)!(2s+1)\{s(s+1)\}^k} \right]^{\frac{1}{2}}. \tag{3.33}$$

It could be easily verified that $F(\theta, \phi)$ is a real, normalised function of θ, ϕ . All the three functions $P(\theta, \phi)$, $Q(\theta, \phi)$, and $F(\theta, \phi)$ have the properties of statistical distribution functions in the sense that, they are normalised to unity and they yield correct expectation values for quantum mechanical spin observables, based on the correspondence rules associating $\hat{\tau}_q^k(\hat{S})$ to spherical harmonic functions $Y_{kq}(\theta, \phi)$, each with different weight factors, given explicitly through,

- P -representation:

$$\hat{\tau}_q^k(\hat{S}) \longrightarrow \sqrt{\frac{4\pi}{(2s+1)}} (2s+1)! \frac{(-1)^{k+q} Y_{kq}(\theta, \phi)}{\sqrt{(2s-k)!(2s+k+1)!}}, \tag{3.34}$$

- Q -representation:

$$\hat{\tau}_q^k(\hat{S}) \longrightarrow \sqrt{\frac{4\pi}{(2s+1)}} \frac{(-1)^{k+q}}{(2s)!} \sqrt{(2s-k)!(2s+k+1)!} Y_{kq}(\theta, \phi), \tag{3.35}$$

- F -representation:

$$\hat{\tau}_q^k(\hat{S}) \longrightarrow 2^k \left[\frac{4\pi(2s+1)(2s-k)!\{s(s+1)\}^k}{(2s+k+1)!} \right]^{\frac{1}{2}} Y_{kq}(\theta, \phi). \tag{3.36}$$

IV. P -, Q -, AND F - FUNCTIONS FOR BIPARTITE QUANTUM SYSTEMS

The P and Q distribution functions can be readily extended for bipartite spin assemblies as follows:

$$\hat{\rho} = \int \int d\Omega_1 d\Omega_2 P(\theta_1, \phi_1; \theta_2, \phi_2) |\theta_1, \phi_1; \theta_2, \phi_2\rangle \langle \theta_1, \phi_1; \theta_2, \phi_2| \tag{4.1}$$

and

$$Q(\theta_1, \phi_1; \theta_2, \phi_2) = \frac{(2s_1+1)(2s_2+1)}{(4\pi)^2} \langle \theta_1, \phi_1; \theta_2, \phi_2 | \hat{\rho} | \theta_1, \phi_1; \theta_2, \phi_2 \rangle \tag{4.2}$$

where $|\theta_1, \phi_1; \theta_2, \phi_2\rangle$ are product spin coherent states with (θ_1, ϕ_1) , (θ_2, ϕ_2) representing angular directions on respective Bloch spheres corresponding to systems with spin s_1 and s_2 . The results of section III could be easily generalised to the case of a pair of systems and we obtain

$$P(\theta_1, \phi_1; \theta_2, \phi_2) = \frac{1}{4\pi} \sum_{k_1=0}^{2s_1} \sum_{q_1=-k_1}^{k_1} \sum_{k_2=0}^{2s_2} \sum_{q_2=-k_2}^{k_2} (-1)^{k_1+q_1} (-1)^{k_2+q_2} \mathcal{P}_{k_1 s_1} \mathcal{P}_{k_2 s_2}$$

$$\times t_{q_1 q_2}^{k_1 k_2} Y_{k_1 q_1}^* (\theta_1, \phi_1) Y_{k_2 q_2}^* (\theta_2, \phi_2) \quad (4.3)$$

and

$$Q(\theta_1, \phi_1; \theta_2, \phi_2) = \frac{1}{4\pi} \sum_{k_1=0}^{2s_1} \sum_{q_1=-k_1}^{k_1} \sum_{k_2=0}^{2s_2} \sum_{q_2=-k_2}^{k_2} (-1)^{k_1+q_1} (-1)^{k_2+q_2} Q_{k_1 s_1} Q_{k_2 s_2} \\ \times t_{q_1 q_2}^{k_1 k_2} Y_{k_1 q_1}^* (\theta_1, \phi_1) Y_{k_2 q_2}^* (\theta_2, \phi_2) \quad (4.4)$$

Let us now consider a EPRB spin- s singlet. The spin density operator in this case is $\hat{\rho} = |(ss)00\rangle \langle (ss)00|$, the matrix elements of which are given by

$$\langle sm'_1; sm'_2 | \hat{\rho} | sm_1; sm_2 \rangle = \frac{(-1)^{m_1-m'_1}}{(2s+1)} \delta_{m'_1-m'_2} \delta_{m_1-m_2}, \quad (4.5)$$

in the $(2s+1) \times (2s+1)$ dimensional direct product spin space of particles 1 and 2. The coupled Fano statistical parameters $t_{q_1 q_2}^{k_1 k_2}$ characterising EPRB spin correlations are given by,

$$t_{q_1 q_2}^{k_1 k_2} = \frac{\text{Tr}[\hat{\rho} \hat{\tau}_{q_1}^{k_1}(\hat{S}_1) \times \hat{\tau}_{q_2}^{k_2}(\hat{S}_2)]}{\sum_{m_1, m'_1, m_2, m'_2=-s}^s \frac{(-1)^{m_1-m'_1}}{(2s+1)} \delta_{m'_1-m'_2} \delta_{m_1-m_2} \langle sm_1 | \hat{\tau}_{q_1}^{k_1}(\hat{S}_1) | sm'_1 \rangle \langle sm_2 | \hat{\tau}_{q_2}^{k_2}(\hat{S}_2) | sm'_2 \rangle} \\ = \sum_{m_1, m'_1=-s}^s \frac{(-1)^{m_1-m'_1}}{(2s+1)} \langle sm_1 | \hat{\tau}_{q_1}^{k_1}(\hat{S}_1) | sm'_1 \rangle \langle s-m_1 | \hat{\tau}_{q_2}^{k_2}(\hat{S}_2) | s-m'_1 \rangle \\ = \sum_{m_1, m'_1=-s}^s \frac{(-1)^{m_1-m'_1}}{(2s+1)} c(ks; m'_1 q_1 m_1) c(ks; -m'_1 q_1 - m_1) \sqrt{(2k_1+1)(2k_2+1)} \\ = (-1)^{k_1+q_1} \delta_{k_1 k_2} \delta_{q_1 - q_2} \quad (4.6)$$

where we have made use of Eqs. (2.2), (2.7) and (2.8)). Thus, the P and Q functions assume the simple form

$$P(\theta_1, \phi_1; \theta_2, \phi_2) = \frac{1}{4\pi} \sum_{k=0}^{2s} (-1)^k \mathcal{P}_{sk}^2 \sum_{q=-k}^k Y_{kq}^*(\theta_1, \phi_1) Y_{kq}(\theta_2, \phi_2) \quad (4.7)$$

and

$$Q(\theta_1, \phi_1; \theta_2, \phi_2) = \frac{1}{4\pi} \sum_{k=0}^{2s} (-1)^k Q_{sk}^2 \sum_{q=-k}^k Y_{kq}^*(\theta_1, \phi_1) Y_{kq}(\theta_2, \phi_2) \quad (4.8)$$

for EPRB spin correlations.

The F representation can be generalised for pairs of vector statistical variates \vec{X}_1 , \vec{X}_2 denoting '*classical spin vectors*' constrained by the conditions $|\vec{X}_1| = \sqrt{s_1(s_1+1)}$, $|\vec{X}_2| = \sqrt{s_2(s_2+1)}$ and we obtain, on using the method outlined in Section III,

$$F(\vec{X}_1, \vec{X}_2) = \delta(\sqrt{s_1(s_1+1)} - |\vec{X}_1|) \sum_{k_1=0}^{2s_1} \frac{1}{k_1! \mathcal{N}_{k_1 s_1}} \{s_1(s_1+1)\}^{\frac{-k_1-2}{2}} \\ \times \delta(\sqrt{s_2(s_2+1)} - |\vec{X}_2|) \sum_{k_2=0}^{2s_2} \frac{1}{k_2! \mathcal{N}_{k_2 s_2}} \{s_2(s_2+1)\}^{\frac{-k_2-2}{2}} \\ \times \sum_{q_1=-k_1}^{k_1} \sum_{q_2=-k_2}^{k_2} t_{q_1 q_2}^{k_1 k_2} Y_{k_1 q_1}^* (\theta_1, \phi_1) Y_{k_2 q_2}^* (\theta_2, \phi_2) \quad (4.9)$$

and the normalised distribution of angular variables i.e., $F(\theta_1, \phi_1; \theta_2, \phi_2)$ is obtained through,

$$F(\theta_1, \phi_1; \theta_2, \phi_2) = \int_0^\infty \int_0^\infty X_1^2 dX_1 X_2^2 dX_2 F(\vec{X}_1, \vec{X}_2)$$

$$= \sum_{k_1=0}^{2s_1} \mathcal{F}_{s_1 k_1} \sum_{k_2=0}^{2s_2} \mathcal{F}_{s_2 k_2} \sum_{q_1=-k_1}^{k_1} \sum_{q_2=-k_2}^{k_2} t_{q_1 q_2}^{k_1 k_2} Y_{k_1 q_1}^*(\theta_1, \phi_1) Y_{k_2 q_2}^*(\theta_2, \phi_2). \quad (4.10)$$

For EPRB spin correlations $F(\theta_1, \phi_1; \theta_2, \phi_2)$ reduces to

$$F(\theta_1, \phi_1; \theta_2, \phi_2) = \frac{1}{4\pi} \sum_{k=0}^{2s} (-1)^k \mathcal{F}_{sk}^2 \sum_{q=-k}^k Y_{kq}^*(\theta_1, \phi_1) Y_{kq}(\theta_2, \phi_2). \quad (4.11)$$

Normalisation property of these functions follows naturally from the orthogonality of the spherical harmonics. It could be verified that the marginal distributions for system 1 in each of these representations is given by

$$\begin{aligned} P(\theta_1, \phi_1) &= \int d\Omega_1 P(\theta_1, \phi_1; \theta_2, \phi_2) = \frac{1}{4\pi}, \\ Q(\theta_1, \phi_1) &= \int d\Omega_1 Q(\theta_1, \phi_1; \theta_2, \phi_2) = \frac{1}{4\pi}, \\ F(\theta_1, \phi_1) &= \int d\Omega_1 F(\theta_1, \phi_1; \theta_2, \phi_2) = \frac{1}{4\pi}, \end{aligned} \quad (4.12)$$

where $d\Omega_1 = \sin \theta_1 d\theta_1 d\phi_1$. Similarly, for system 2, the marginal distributions are realised to be $P(\theta_2, \phi_2) = \frac{1}{4\pi}$; $Q(\theta_2, \phi_2) = \frac{1}{4\pi}$; and $F(\theta_2, \phi_2) = \frac{1}{4\pi}$. These marginal distributions are spherically symmetric and hence correspond to totally random orientations (θ_1, ϕ_1) and (θ_2, ϕ_2) of the spin vectors associated with the system.

The spin correlations $\langle (\hat{\vec{S}}_1 \cdot \vec{a})(\hat{\vec{S}}_2 \cdot \vec{b}) \rangle$ could be evaluated using the above distribution functions as follows: We make use of the relations between irreducible tensors of rank 1 and the spin components $\hat{S}_1, \hat{S}_2, \hat{S}_3$, given by

$$\begin{aligned} \hat{S}_1 &= \sqrt{\frac{s(s+1)}{6}} \left(\hat{\tau}_{-1}^1(\hat{\vec{S}}) - \hat{\tau}_1^1(\hat{\vec{S}}) \right) \\ \hat{S}_2 &= i\sqrt{\frac{s(s+1)}{6}} \left(\hat{\tau}_{-1}^1(\hat{\vec{S}}) + \hat{\tau}_1^1(\hat{\vec{S}}) \right) \\ \hat{S}_3 &= \sqrt{\frac{s(s+1)}{3}} \hat{\tau}_0^1(\hat{\vec{S}}) \end{aligned} \quad (4.13)$$

which lead to the correspondence rules (see Eqs.(3.34-3.36) of Section III) for the spin operator $\hat{\vec{S}}$ in the three representations discussed above, as

$$\begin{aligned} P - \text{representation} : \quad \hat{\vec{S}} &\longrightarrow s \vec{n}(\pi - \theta, \phi), \\ Q - \text{representation} : \quad \hat{\vec{S}} &\longrightarrow (s+1) \vec{n}(\pi - \theta, \phi), \\ F - \text{representation} : \quad \hat{\vec{S}} &\longrightarrow \sqrt{s(s+1)} \vec{n}(\theta, \phi) \end{aligned} \quad (4.14)$$

where \vec{n} denotes a unit vector specified by the polar co-ordinates in the arguments. Thus, we have

$$\begin{aligned} \langle (\hat{\vec{S}}_1 \cdot \vec{a})(\hat{\vec{S}}_2 \cdot \vec{b}) \rangle &= s^2 \int \int d\Omega_1 d\Omega_2 P(\theta_1, \phi_1; \theta_2, \phi_2) (\vec{n}_1 \cdot \vec{a}) (\vec{n}_2 \cdot \vec{b}) \\ &= (s+1)^2 \int \int d\Omega_1 d\Omega_2 Q(\theta_1, \phi_1; \theta_2, \phi_2) (\vec{n}_1 \cdot \vec{a}) (\vec{n}_2 \cdot \vec{b}) \\ &= \{s(s+1)\}^2 \int \int d\Omega_1 d\Omega_2 F(\theta_1, \phi_1; \theta_2, \phi_2) (\vec{n}_1 \cdot \vec{a}) (\vec{n}_2 \cdot \vec{b}) \end{aligned} \quad (4.15)$$

which, on expressing $\vec{n} \cdot \vec{a} = \frac{4\pi}{3} \sum_{q=-1}^1 Y_{1q}(\vec{n}) Y_{1q}^*(\vec{a})$ and on using the orthonormality property of spherical harmonics, yields the well-known result for EPRB spin- s correlations,

$$\langle (\hat{\vec{S}}_1 \cdot \vec{a})(\hat{\vec{S}}_2 \cdot \vec{b}) \rangle = -\frac{s(s+1)}{3} \cos \theta_{ab}. \quad (4.16)$$

Here, $\cos \theta_{ab} = \vec{a} \cdot \vec{b}$.

One can make use of the addition theorem for spherical harmonics[11]

$$\sum_q Y_{kq}(\theta_1, \phi_1) Y_{kq}^*(\theta_2, \phi_2) = \frac{2k+1}{4\pi} P_k(\cos \theta_{12}), \quad (4.17)$$

where P_k denote Legendre polynomial of order k ; $\cos \theta_{12} = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2)$, so that the spin distributions reduce to

$$\begin{aligned} P(\theta_{12}) &= \frac{1}{(4\pi)^2} \sum_{k=0}^{2j} (-1)^k (2k+1) \mathcal{P}_{sk}^2 P_k(\cos \theta_{12}) \\ Q(\theta_{12}) &= \frac{1}{(4\pi)^2} \sum_{k=0}^{2j} (-1)^k (2k+1) \mathcal{Q}_{sk}^2 P_k(\cos \theta_{12}) \\ F(\theta_{12}) &= \frac{1}{(4\pi)^2} \sum_{k=0}^{2j} (-1)^k (2k+1) \mathcal{F}_{sk}^2 P_k(\cos \theta_{12}). \end{aligned} \quad (4.18)$$

We give below the P , Q , and F functions explicitly for the spin- $\frac{1}{2}$ case:

$$\begin{aligned} P^{\frac{1}{2}}(\theta_{12}) &= \frac{1}{(4\pi)^2} (1 - 9 \cos \theta_{12}) \\ Q^{\frac{1}{2}}(\theta_{12}) &= \frac{1}{(4\pi)^2} (1 - \cos \theta_{12}) \\ f^{\frac{1}{2}}(\theta_{12}) &= \frac{1}{(4\pi)^2} (1 - 3 \cos \theta_{12}) \end{aligned} \quad (4.19)$$

where $\cos \theta_{12} = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2)$. In Fig.1 we have plotted P , Q , and F functions for spin values $s = \frac{1}{2}, 1, \frac{3}{2}, 2$. It could be observed that the distributions show prominent peaks around $\theta_{12} = 180^\circ$ indicating the anticorrelation property of associated classical vectors \vec{n}_1 and \vec{n}_2 .

Expressing $(2s \pm k)! = (2s)^{2s \pm k} \prod_{n=\mp k}^{2s-1} \left(1 - \frac{n}{2s}\right)$ etc., in \mathcal{P}_{sk} , \mathcal{Q}_{sk} and \mathcal{F}_{sk} it can be readily observed that

$$\lim_{s \rightarrow \infty} \mathcal{P}_{sk} = 1, \quad \lim_{s \rightarrow \infty} \mathcal{Q}_{sk} = 1, \quad \lim_{s \rightarrow \infty} \mathcal{F}_{sk} = 1, \quad (4.20)$$

which, together with the completeness property[11]

$$\sum_{k=0}^{\infty} \sum_{q=-k}^k Y_{kq}^*(\theta, \phi) Y_{kq}(\theta', \phi') = \delta(\cos \theta - \cos \theta') \delta(\phi - \phi'), \quad (4.21)$$

and the symmetry

$$Y_{kq}(\theta, \phi) = (-1)^k Y_{kq}(\pi - \theta, \pi + \phi) \quad (4.22)$$

of the spherical harmonic functions, leads to the P , Q , and F distribution functions in the classical limit as,

$$\begin{aligned} \lim_{s \rightarrow \infty} P(\theta_1, \phi_1; \theta_2, \phi_2) &= \frac{1}{4\pi} \delta(\cos \theta_1 + \cos \theta_2) \delta(\phi_1 - (\phi_2 + \pi)), \\ \lim_{s \rightarrow \infty} Q(\theta_1, \phi_1; \theta_2, \phi_2) &= \frac{1}{4\pi} \delta(\cos \theta_1 + \cos \theta_2) \delta(\phi_1 - (\phi_2 + \pi)), \\ \lim_{s \rightarrow \infty} F(\theta_1, \phi_1; \theta_2, \phi_2) &= \frac{1}{4\pi} \delta(\cos \theta_1 + \cos \theta_2) \delta(\phi_1 - (\phi_2 + \pi)). \end{aligned} \quad (4.23)$$

Observe that in the classical limit, these distribution functions reflect the perfect anticorrelation between the classical vectors \vec{n}_1 and \vec{n}_2 .

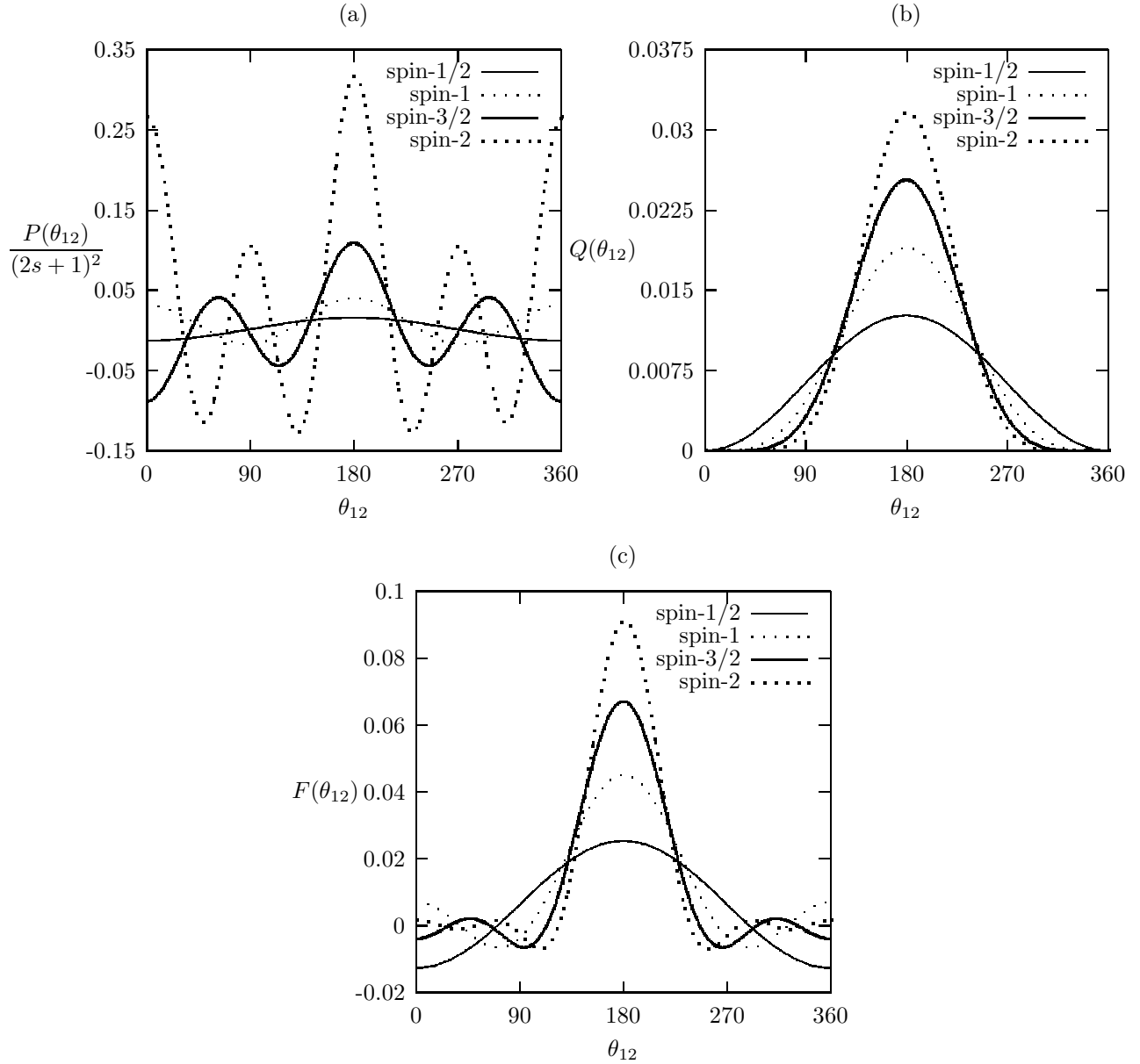


FIG. 1. Spin distribution functions (a) $P(\theta_{12})$, (b) $Q(\theta_{12})$ and (c) $F(\theta_{12})$ as a function of the angle θ_{12} between the classical spin vectors \vec{n}_1 and \vec{n}_2 constituting the spin singlet.

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- [17] The factor $\frac{1}{k! \mathcal{N}_{sk}}$ appears because $\hat{\tau}_q^k(\hat{S})$ are nothing but symmetrised solid harmonics involving \hat{S} with a normalisation factor[9] $\frac{1}{k! \mathcal{N}_{sk}}$.